How probable is an infinite sequence of heads? A reply to Williamson

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It is possible that a fair coin tossed infinitely many times will always land heads. So the probability of such a sequence of outcomes should, intuitively, be positive, albeit miniscule: 0 probability ought to be reserved for *impossible* events. And furthermore, since the tosses are independent and the probability of heads (and tails) on a single toss is $^1/_2$, all sequences are equiprobable. Yet familiarly, it is not possible to assign all sequences the same positive (real-numbered) probability.[1] So either all sequences will be assigned 0 probability, or they will not all have the same probability. Real-valued probabilities reflect our probabilistic intuitions imperfectly.

In non-standard analysis (Bernstein and Wattenberg 1969), by way of contrast, it seems as if our probabilistic intuitions may be adequately reflected. Here, where the numbers include infinitesimals, larger than 0 yet

[1] If there are non-denumerably many disjoint events in a probability space, all but denumerably many of them must be assigned probability 0. Here is a simple proof: Suppose to the contrary, and let X be a non-denumerable set of disjoint events with positive probabilities. Define: $A_n = \{x\varepsilon X: P\{x\}>1/n\}$. Now, $UA_n = \{x\varepsilon X: P\{x\} > 0\}$. By the assumption, UA_n is non-denumerable, and thus, for some n, A_n is non-denumerable, and hence has a subset B whose cardinality is n. B is a subset of X, so its elements are disjoint. By additivity, P(B) is greater than 1, contrary to the supposition that P is a probability.

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smaller than any positive real number, the impossibility proof fails,[2] and probabilities may be assigned so that the tosses are equiprobable and each has a positive probability.

Our sanguinity, Williamson claims (2007), is unjustified. He adduces an argument which purports to show that the probability of every infinite sequence must be 0, the possibility of assigning to them all the same positive (infinitesimal) probability notwithstanding.

Let H(1...) be the event that every toss comes up heads, H(2...) the event that every toss after the first comes up heads, and H(1) the event that the first toss comes up heads. Now, H(1...) is the conjunction of H(2...) and H(1), and since the tosses are independent, P(H(1...) = $^1/_2 \cdot$P(H(2...). But equally, P(H(1...) = P(H(2...), since the two events H(1...) and H(2...) are 'isomorphic ... they differ only in the *inconsequential* respect that H(2...) starts one second after H(1...). That H(2...) is preceded by another toss is *irrelevant*, given the independence of the tosses. Thus H(1...) and H(2...) should have the same probability' (2007: 175, my italics). Putting these two claims together, we get the result that P(H(2...)) = $^1/_2 \cdot$P(H(2)). But in non-standard analysis, too, if $x = {}^1/_2 \cdot x$, $x = 0$. So, Williamson concludes, P(H(2...)) = P(H(1...)) = 0, and non-standard analysis is of no avail: (some of) our probabilistic intuitions concerning the coin-tosses cannot be adequately reflected in it, either.

'What has gone wrong?' Williamson wonders (2007: 180), and his diagnosis is that 'some natural, apparently compelling forms of reasoning [pertaining to probabilities] fail for infinite sets' (2007: 180). He draws an analogy between the probabilistic case and set-theory, in which, for instance, a set and its proper subset may (very unintuitively) have the same cardinality ('size'). The set of natural numbers, for instance, has the same cardinality as the set of even numbers, indeed of the (much sparser) set of prime numbers.

It is Williamson's pessimism which is unwarranted, I contend. His argument for the gloomy conclusion is specious. The culprit is the argument's second premiss. (The first one is a trivial invocation of the probability calculus.) *Pace* Williamson, I will argue, isomorphism (of sequences of events) isn't a plausible sufficient condition for equiprobability.

Two sequences of events are 'isomorphic' if we can 'map the constituent events of the first one-one onto the constituent events of the second in a natural way that preserves the physical structure of the set-up' (2007: 175). And isomorphism is sufficient for equiprobability, Williamson

[2] The invocation of additivity to the subset whose cardinality is n and all of whose members have the same positive probability doesn't here yield the (non-probabilistic) claim that the probability of the union is greater than 1.

claims, because physical chances supervene on more basic physical properties, and these are preserved by isomorphism. But in fact, Williamson's example shows that isomorphism doesn't preserve *all* basic physical properties. He claims that the two 'sequences of events are of *exactly* the same qualitative type' (2007: 178, original italics). But although all the physical properties of the *constituent events* are preserved by the mapping, as are the temporal intervals between adjacent tosses, there is a *global* property (of the complex event) which is *not* preserved. The second sequence is a *proper subset* of the first. So they are not physically identical in a way which would allow us to invoke supervenience and infer that they are equiprobable.

To 'make the point more vivid' (2007: 175), Williamson couches the argument in terms of two sequences involving *two* coins. '[T]hat the first coin will be tossed once before the H(2...) sequence begins is *irrelevant*', he argues (2007: 175–76, my italics). But in fact, it makes all the difference. Here, the two sequences are disjoint, so we cannot object – as before – that one sequence is a proper subset of the other. But a similar objection can be invoked. The *set of temporal points occupied by one sequence* is a proper subset of those occupied by the second sequence. So the two sequences do not share *all* their physical properties. And this serves – yet again – to show that isomorphism, despite its seeming stringency, isn't sufficient for equiprobability, thus blocking the paradoxical reasoning.

We can now dispel another mystery Williamson (2007: 179) thinks he has discerned. We prefer a lottery in which we win a prize if H(1...) occurs over one in which we win (the same prize) if H(2...) occurs, and this preference seems rational: in the first lottery, there is an additional hurdle to surmount before we can claim the prize, and the probability of failure is $^1/_2$. So the (subjective) probability of winning in the second lottery must (rationally) be larger. But the paradoxical reasoning purports to show that we ought to be *indifferent*. According to Lewis's (1980) plausible 'Principal Principle', if the only thing of relevance we know is an event's physical chance, that ought to be the credence we assign to it occurring. And this engenders a dilemma with respect to rational credences. We seemingly have to choose between the Principal Principle and our intuitions about the two bets.

The dilemma can be dispelled. It is only the paradoxical – and now discredited – reasoning that leads, via the Principal Principle, to the unintuitive conclusion about the two bets.[3]

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References

Bernstein, A. R. and F. Wattenberg. 1969. Non-standard measure theory. In *Applications of Model Theory of Algebra, Analysis, and Probability*, ed. W. A. J. Luxemburg. New York: Holt, Rinehard and Winston.

Lewis, D. 1980. A subjectivist's guide to objective probability. In *Ifs: Conditionals, Belief, Decision, Chance and Time*, ed. W. L. Harper, R. Stalnaker and G. Pearce, 1981: 267–97. Boston: Reidel.

Williamson, T. 2007. How probable is an infinite sequence of heads? *Analysis* 67: 173–80.

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